

Generalized Heisenberg's Dynamics

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(Received)

We formulate a dynamical system based on many-index objects. These objects yield a generalization of the Heisenberg's equation. Systems describing harmonic oscillators are given.

§1. Introduction

Recently, a new mechanics has been proposed that is based on three-index objects¹⁾, and its basic structure has been studied from an algebraic point of view^{2), 3)}. This mechanics has a counterpart in the canonical structure of classical mechanics or Nambu mechanics⁴⁾, and can be interpreted as its 'quantum' or 'discretized' version. It can also be regarded as a generalization of Heisenberg's matrix mechanics, because a generalization of the Ritz rule and that of the Bohr's frequency condition are employed as guiding principles.

The mechanics, in which physical variables are n -index objects ($n \geq 4$), was also proposed in Ref. 1), but its formulation has not yet been completed. The purpose of the present paper is the construction of a mechanics for mutli-index objects, modeling the dynamical structure of Heisenberg's matrix mechanics.

This paper is organized as follows. In the next section, we explain the definition of n -index objects. We formulate a dynamical system based on n -index objects in §3. Section 4 is devoted to conclusions.

§2. Generalized matrices

2.1. Definitions

We state our definition of n -index objects (we refer to them as n -th power matrices)^{**)} and define related terminology. An n -th power matrix is an object with n indices written $A_{l_1 l_2 \dots l_n}$, which is a generalization of a usual matrix written analogously as $B_{l_1 l_2}$. We handle n -th power matrices that every line has a same size, i.e., $N \times N \times \dots \times N$ matrices, and treat the elements of an n -th power matrix as c -numbers throughout this paper.

First we define the hermiticity of an n -th power matrix by the relation $A_{l'_1 l'_2 \dots l'_n} = A_{l_1 l_2 \dots l_n}^*$ for odd permutations among indices and refer to an n -th power matrix possessing hermiticity as a hermitian n -th power matrix. Here, the asterisk indicates

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^{**)} Many-index objects have been introduced to construct a quantum version of the Nambu bracket.^{5), 6)} The definition of the n -fold product we use is the same as that by Xiong.

complex conjugation. A hermitian n -th power matrix has the relation $A_{l'_1 l'_2 \dots l'_n} = A_{l_1 l_2 \dots l_n}$ for even permutations among indices. The components with a same index, e.g., $A_{l_1 \dots l_i \dots l_i \dots l_n}$, which is a counterpart of a diagonal part in a hermitian matrix, are real-valued and symmetric with respect to permutations among indices $\{l_1, \dots, l_i, \dots, l_i, \dots, l_n\}$. We refer to a special type of hermitian matrix whose components with completely different indices are vanishing as a real normal form or a real normal n -th power matrix. An normal n -th power matrix is written

$$A_{l_1 l_2 \dots l_n}^{(N)} = \sum_{i < j} \delta_{l_i l_j} a_{l_j l_1 \dots \hat{l}_i \dots \hat{l}_j \dots l_n}, \quad (2.1)$$

where the summation is over all pairs among $\{l_1, \dots, l_n\}$, the indices with a hat are omitted, and $a_{l_j l_1 \dots \hat{l}_i \dots \hat{l}_j \dots l_n}$ is symmetric under the exchange of $n - 2$ indices $\{l_1, \dots, \hat{l}_i, \dots, \hat{l}_j, \dots, l_n\}$.

We define the n -fold product of n -th power matrices $(A_i)_{l_1 l_2 \dots l_n}$, ($i = 1, 2, \dots, n$) by

$$(A_1 A_2 \dots A_n)_{l_1 l_2 \dots l_n} \equiv \sum_k (A_1)_{l_1 \dots l_{n-1} k} (A_2)_{l_1 \dots l_{n-2} k l_n} \dots (A_n)_{k l_2 \dots l_n}. \quad (2.2)$$

The resultant n -index object, $(A_1 A_2 \dots A_n)_{l_1 l_2 \dots l_n}$, does not necessarily possess hermiticity, even if $(A_i)_{l_1 l_2 \dots l_n}$ s are hermitian n -th power matrices. Note that the above product is, in general, neither commutative nor associative; for example,

$$\begin{aligned} (A_1 A_2 \dots A_n)_{l_1 l_2 \dots l_n} &\neq (A_2 A_1 \dots A_n)_{l_1 l_2 \dots l_n}, \\ (A_1 \dots A_{n-1} (A_n A_{n+1} \dots A_{2n-1}))_{l_1 l_2 \dots l_n} \\ &\neq ((A_1 \dots A_{n-1} A_n) A_{n+1} \dots A_{2n-1})_{l_1 l_2 \dots l_n}. \end{aligned} \quad (2.3)$$

The n -fold commutator is defined by

$$\begin{aligned} [A_1, A_2, \dots, A_n]_{l_1 l_2 \dots l_n} \\ \equiv \sum_{(i_1, \dots, i_n)} \sum_k \text{sgn}(P) (A_{i_1})_{l_1 \dots l_{n-1} k} (A_{i_2})_{l_1 \dots l_{n-2} k l_n} \dots (A_{i_n})_{k l_2 \dots l_n}, \end{aligned} \quad (2.4)$$

where the first summation is over all permutations among the subscripts $\{i_1, \dots, i_n\}$. Here, $\text{sgn}(P)$ is $+1$ and -1 for even and odd permutations among the subscripts $\{i_1, \dots, i_n\}$, respectively. If $(A_i)_{l_1 l_2 \dots l_n}$ s are hermitian n -th power matrices, then $i[A_1, A_2, \dots, A_n]_{l_1 l_2 \dots l_n}$ is also hermitian.

2.2. Properties

We study some properties on the n -fold commutator $[A_1, A_2, \dots, A_n]_{l_1 l_2 \dots l_n}$. This commutator is written

$$\begin{aligned} [A_1, A_2, \dots, A_n]_{l_1 l_2 \dots l_n} &= (A_1)_{l_1 l_2 \dots l_n} (A_2 \widetilde{A_3 \dots A_n})_{l_1 l_2 \dots l_n} \\ &\quad + (-1)^{n-1} (A_2)_{l_1 l_2 \dots l_n} (A_3 \dots \widetilde{A_n A_1})_{l_1 l_2 \dots l_n} \\ &\quad + \dots + (-1)^{n-1} (A_n)_{l_1 l_2 \dots l_n} (A_1 A_2 \dots \widetilde{A_{n-1}})_{l_1 l_2 \dots l_n} \\ &\quad + ([A_1, A_2, \dots, A_n])_{l_1 l_2 \dots l_n}^0, \end{aligned} \quad (2.5)$$

where $(A_2 \widetilde{A_3 \cdots A_n})_{l_1 l_2 \cdots l_n}$ and $([A_1, A_2, \cdots, A_n])_{l_1 l_2 \cdots l_n}^0$ are defined by

$$\begin{aligned} & (A_2 \widetilde{A_3 \cdots A_n})_{l_1 l_2 \cdots l_n} \\ & \equiv \sum_{(i_2, \dots, i_n)} \text{sgn}(P) \left((A_{i_2})_{l_1 \cdots l_{n-2} l_n} (A_{i_3})_{l_1 \cdots l_{n-3} l_n l_{n-1} l_n} \cdots (A_{i_n})_{l_n l_2 \cdots l_{n-1} l_n} \right. \\ & \quad + (-1)^{n-1} (A_{i_2})_{l_1 \cdots l_{n-3} l_{n-1} l_{n-1} l_n} (A_{i_3})_{l_1 \cdots l_{n-4} l_{n-1} l_{n-2} l_{n-1} l_n} \cdots (A_{i_n})_{l_1 \cdots l_{n-2} l_{n-1} l_{n-1} l_n} \\ & \quad \left. + \cdots + (-1)^{n-1} (A_{i_2})_{l_1 \cdots l_{n-1} l_1} (A_{i_3})_{l_1 \cdots l_{n-2} l_1 l_n} \cdots (A_{i_n})_{l_1 l_1 l_3 \cdots l_n} \right) \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} & ([A_1, A_2, \cdots, A_n])_{l_1 l_2 \cdots l_n}^0 \\ & \equiv \sum_{(i_1, \dots, i_n)} \sum_{k \neq l_1, l_2, \dots, l_n} \text{sgn}(P) (A_{i_1})_{l_1 \cdots l_{n-1} k} (A_{i_2})_{l_1 \cdots l_{n-2} k l_n} \cdots (A_{i_n})_{k l_2 \cdots l_n} \end{aligned} \quad (2.7)$$

respectively.

We discuss features of $(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 l_2 \cdots l_n}$. It possesses skew-symmetry with respect to permutations among indices:

$$(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 \cdots l_i \cdots l_j \cdots l_n} = -(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 \cdots l_j \cdots l_i \cdots l_n}, \quad (2.8)$$

if $(A_k)_{l_j l_1 \cdots \hat{l}_i \cdots l_n}$ s, $(k = 1, \dots, n-1)$ are symmetric with respect to permutations among n -indices $\{l_j, l_1, \dots, \hat{l}_i, \dots, l_n\}$ as hermitian n -th power matrices are. Let us define an operation for k -th anti-symmetric objects $\omega_{l_1 l_2 \cdots l_k}$ by

$$(\delta \omega)_{l_0 l_1 l_2 \cdots l_k} \equiv \sum_{i=0}^k (-1)^i \omega_{l_0 l_1 \cdots \hat{l}_i \cdots l_k}. \quad (2.9)$$

Here the operator δ is regarded as a coboundary operator that changes k -th anti-symmetric objects into $(k+1)$ -th objects, and this operation is nilpotent, i.e. $\delta^2(*) = 0$.* If the $\omega_{l_1 l_2 \cdots l_k}$ satisfies the cocycle condition: $(\delta \omega)_{l_0 l_1 l_2 \cdots l_k} = 0$, it is called a k -cocycle. We give an example of a solution for the cocycle condition: $(\delta(A_1 \widetilde{A_2 \cdots A_{n-1}}))_{l_0 l_1 l_2 \cdots l_n} = 0$. When one of A_l s is an arbitrary hermitian n -th power matrix and all the rest have components in the form $(A_l)_{l_j l_1 \cdots \hat{l}_i \cdots l_n} = \sum_{l_k \neq \hat{l}_i} (a_l)_{l_k}$, the $(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 l_2 \cdots l_n}$ is written

$$(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 l_2 \cdots l_n} = \sum_{i=1}^n (-1)^{i-1} \gamma_{l_1 \cdots \hat{l}_i \cdots l_n} \equiv (\delta \gamma)_{l_1 l_2 \cdots l_n}, \quad (2.10)$$

where $\gamma_{l_1 l_2 \cdots l_{n-1}}$ is an $(n-1)$ -th antisymmetric objects. Then, the n -th antisymmetric object $(A_1 \widetilde{A_2 \cdots A_{n-1}})_{l_1 l_2 \cdots l_n}$ automatically satisfies the cocycle condition:

$$(\delta(A_1 \widetilde{A_2 \cdots A_{n-1}}))_{l_0 l_1 l_2 \cdots l_n} = (\delta^2 \gamma)_{l_0 l_1 \cdots \hat{l}_i \cdots l_n} = 0 \quad (2.11)$$

*) See Ref. 7) for textbooks with respect to cohomology.

due to the nilpotency for coboundary operations. This type of solution is called an n -coboundary.

We can show the following relations on the n -fold commutator from the above expressions and properties.

1. For arbitrary n -th power hermitian matrices A_j , $[A_1, \dots, A_{n-1}, \Delta]_{l_1 l_2 \dots l_n} = 0$ with $\Delta_{l_1 l_2 \dots l_n} = \prod_{i < j} \delta_{l_i l_j}$. Here the product is over all pairs among indices.
2. For arbitrary normal n -th power matrices $A_j^{(N)}$, the n -fold commutator among A and $A_j^{(N)}$ is given by

$$[A, A_1^{(N)}, \dots, A_{n-1}^{(N)}]_{l_1 l_2 \dots l_n} = A_{l_1 l_2 \dots l_n} (A_1^{(N)} \cdots A_{n-1}^{(N)})_{l_1 l_2 \dots l_n}. \quad (2 \cdot 12)$$

3. The n -fold commutator among arbitrary normal n -th power matrices $A_i^{(N)}$ is vanishing;

$$[A_1^{(N)}, A_2^{(N)}, \dots, A_n^{(N)}]_{l_1 l_2 \dots l_n} = 0. \quad (2 \cdot 13)$$

4. If $(B_1^{(N)} B_2^{(N)} \cdots B_{n-1}^{(N)})_{l_1 l_2 \dots l_n}$ is an n -cocycle for normal n -th power matrices $B_j^{(N)}$, the fundamental identity holds:

$$\begin{aligned} & [[A_1, \dots, A_n], B_1^{(N)}, \dots, B_{n-1}^{(N)}]_{l_1 l_2 \dots l_n} \\ &= \sum_{i=1}^n [A_1, \dots, [A_i, B_1^{(N)}, \dots, B_{n-1}^{(N)}], \dots, A_n]_{l_1 l_2 \dots l_n}. \end{aligned} \quad (2 \cdot 14)$$

§3. Dynamical system

In this section, we employ a generalization of the Ritz rule and that of the Bohr's frequency condition as guiding principles, and construct a generalization of Heisenberg's matrix mechanics based on hermitian n -th power matrices.

3.1. Framework

The time-dependent variables are hermitian n -th power matrices given by

$$(V_\alpha(t))_{l_1 l_2 \dots l_n} = (V_\alpha)_{l_1 l_2 \dots l_n} e^{i\Omega_{l_1 l_2 \dots l_n} t}, \quad (3 \cdot 1)$$

where the angular frequency $\Omega_{l_1 l_2 \dots l_n}$ has the properties

$$\Omega_{l'_1 l'_2 \dots l'_n} = \text{sgn}(P) \Omega_{l_1 l_2 \dots l_n}, \quad (3 \cdot 2)$$

$$(\delta\Omega)_{l_0 l_1 l_2 \dots l_n} \equiv \sum_{i=0}^n (-1)^i \Omega_{l_0 l_1 \dots \hat{l}_i \dots l_n} = 0. \quad (3 \cdot 3)$$

The $\Omega_{l_1 l_2 \dots l_n}$ is regarded as an n -cocycle, from the equation (3.3). This equation is a generalization of the Ritz rule,^{*)} and it is required from a consistency for

^{*)} The Ritz rule is given by $\Omega_{l_1 l_3} = \Omega_{l_1 l_2} + \Omega_{l_2 l_3}$ in quantum mechanics, where $\Omega_{l_i l_j}$ is the angular frequency of radiation from an atom.

the time evolution of a system as will be shown. Notice that the n -fold product, $(V_{\alpha_1} V_{\alpha_2} \cdots V_{\alpha_n})_{l_1 l_2 \cdots l_n} e^{i\Omega_{l_1 l_2 \cdots l_n} t}$, takes the same form as (3.1), with the relation (3.3).

The time-independent variables are real normal n -th power matrices given by

$$(U_a)_{l_1 l_2 \cdots l_n} = \sum_{i < j} \delta_{l_i l_j} (u_a)_{l_j l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_n}. \quad (3.4)$$

These variables are conserved quantities, and are regarded as generators of a symmetry transformation.

Next we discuss the time evolution of physical variables. It is given as a symmetry transformation if the physical system is closed. In our mechanics, it is expected that real normal n -th power matrices generate such a transformation. We refer to them as 'Hamiltonians' H_A ($A = 1, \dots, n-1$). Hamiltonians are functions of physical variables: $H_A = H_A(V_\alpha(t), U_a)$. By analogy with Heisenberg's matrix mechanics, we require that the $(V_\alpha(t))_{l_1 l_2 \cdots l_n}$ s should yield the generalization of the Heisenberg equation:

$$\frac{d}{dt}(V_\alpha(t))_{l_1 l_2 \cdots l_n} = \frac{1}{i\hbar^{(n)}} [V_\alpha(t), H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n}, \quad (3.5)$$

where $\hbar^{(n)}$ is a new physical constant. The left-hand side of (3.5) is written

$$\frac{d}{dt}(V_\alpha(t))_{l_1 l_2 \cdots l_n} = i\Omega_{l_1 l_2 \cdots l_n} (V_\alpha(t))_{l_1 l_2 \cdots l_n}, \quad (3.6)$$

by definition (3.1). On the other hand, the right-hand side is written

$$\begin{aligned} & \frac{1}{i\hbar^{(n)}} [V_\alpha(t), H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} \\ &= \frac{1}{i\hbar^{(n)}} (H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n} (V_\alpha(t))_{l_1 l_2 \cdots l_n}, \end{aligned} \quad (3.7)$$

by use of the formula (2.12). From equations (3.6) and (3.7), we obtain the relation

$$\hbar^{(n)} \Omega_{l_1 l_2 \cdots l_n} = -(H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n}. \quad (3.8)$$

This relation is a generalization of Bohr's frequency condition.*)

Let us make a consistency check for the time evolution of a system. By definition, an arbitrary normal n -th power matrix $A^{(N)}$ (and the time-independent part of $V_\alpha(t)$) should be a constant of motion, and it is verified by use of the equation of motion:

$$\frac{d}{dt}(A^{(N)})_{l_1 l_2 \cdots l_n} = \frac{1}{i\hbar^{(n)}} [A^{(N)}, H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} = 0, \quad (3.9)$$

where the formula (2.13) is used. Since the Hamiltonians are real normal n -th power matrices, they are conserved quantities. The n -fold commutator, $i[V_1(t), \cdots, V_n(t)]$

*) The Bohr's frequency condition is given by $\hbar \Omega_{l_1 l_2} = -\widetilde{H}_{l_1 l_2} = E_{l_1} - E_{l_2}$ in quantum mechanics, where E_l is the energy eigenvalue of an atom.

should satisfy the fundamental identity including the Hamiltonians:

$$\begin{aligned} & [i[V_1(t), \dots, V_n(t)], H_1, \dots, H_{n-1}]_{l_1 l_2 \dots l_n} \\ &= \sum_{i=1}^n i[V_1(t), \dots, [V_i(t), H_1, \dots, H_{n-1}], \dots, V_n(t)]_{l_1 l_2 \dots l_n} \end{aligned} \quad (3.10)$$

from the requirement that the derivation rule for the time should hold such that

$$\frac{d}{dt} i[V_1(t), \dots, V_n(t)]_{l_1 l_2 \dots l_n} = \sum_{i=1}^n i[V_1(t), \dots, \frac{d}{dt} V_i(t), \dots, V_n(t)]_{l_1 l_2 \dots l_n} \quad (3.11)$$

The fundamental identity (3.10) holds in the case that the $\Omega_{l_1 l_2 \dots l_n}$ is an n -cocycle from the 4-th relation on the n -fold commutator.

3.2. Examples

We study the simple example of a harmonic oscillator whose variables are two kinds of hermitian n -th power matrices given by $\xi(t)_{l_1 l_2 \dots l_n} = \xi_{l_1 l_2 \dots l_n} e^{i\Omega_{l_1 l_2 \dots l_n} t}$ and $\eta(t)_{l_1 l_2 \dots l_n} = \eta_{l_1 l_2 \dots l_n} e^{i\Omega_{l_1 l_2 \dots l_n} t}$. Here, each of the indices l_i runs from 1 to n . The coefficients $\xi_{l_1 l_2 \dots l_n}$ and $\eta_{l_1 l_2 \dots l_n}$ are given by

$$\xi_{l_1 l_2 \dots l_n} = \sqrt{\frac{\hbar^{(n)}}{2m\Omega}} |\varepsilon_{l_1 l_2 \dots l_n}|, \quad \eta_{l_1 l_2 \dots l_n} = \frac{1}{i} \sqrt{\frac{m\Omega \hbar^{(n)}}{2}} \varepsilon_{l_1 l_2 \dots l_n}, \quad (3.12)$$

where the quantity m in the square root represents a mass, $\Omega = |\Omega_{12 \dots n}|$ and $\varepsilon_{l_1 l_2 \dots l_n}$ is the n -dimensional Levi-Civita symbol.

If $\Omega_{l_1 l_2 \dots l_n} = -\Omega \varepsilon_{l_1 l_2 \dots l_n}$, we obtain the equations of motion describing the harmonic oscillator:

$$\frac{d}{dt} \xi(t)_{l_1 l_2 \dots l_n} = \frac{1}{m} \eta(t)_{l_1 l_2 \dots l_n}, \quad (3.13)$$

$$\frac{d}{dt} \eta(t)_{l_1 l_2 \dots l_n} = -m\Omega^2 \xi(t)_{l_1 l_2 \dots l_n}. \quad (3.14)$$

The hamiltonians H_A satisfy the following relation

$$\hbar^{(n)} \Omega \varepsilon_{l_1 l_2 \dots l_n} = (H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \dots l_n} \quad (3.15)$$

from the requirement that physical variables should yield the generalized Heisenberg's equation (3.5). As an example of H_{AS} , we have

$$(H_1)_{n \ n \ 1 \dots n-2} = \hbar^{(n)} \Omega, \quad (H_B)_{n \ n \ 1 \dots \hat{i} \dots n-1} = \delta_{n-i} B, \quad (3.16)$$

where H_{AS} are symmetric with respect to permutations among indices, other components of H_{AS} are vanishing, and B runs from 2 to $n-1$. There is the algebraic relation among $\xi(t)$, $\eta(t)$ and H_{AS} :

$$(H_1)_{l_1 l_2 \dots l_n} = i\Omega [\xi(t), \eta(t), H_2, \dots, H_{n-1}]_{l_1 l_2 \dots l_n}. \quad (3.17)$$

Finally we give an example of 'n-plet', $(V_\alpha(t))_{l_1 l_2 \dots l_n}$ ($\alpha = 1, \dots, n$), where each of the indices l_i runs from 1 to n^2 . The components of $V_\alpha(t)$ s are defined by

$$(V_\alpha(t))_{l_1 l_2 \dots l_n} \equiv \begin{cases} \eta(t)_{l_1 l_2 \dots l_n} & \text{for } l_i = (\alpha - 1)n + 1, (\alpha - 1)n + 2, \dots, \alpha n \\ \xi(t)_{l_1 l_2 \dots l_n} & \text{for } l_i = \alpha n + 1, \alpha n + 2, \\ & \dots, (\alpha + 1)n \pmod{n^2} \\ (\zeta_{n-j})_{l_1 l_2 \dots l_n} & \text{for } l_i = (\alpha + j - 1)n + 1, (\alpha + j - 1)n + 2, \\ & \dots, (\alpha + j)n \pmod{n^2}, \end{cases} \quad (3.18)$$

where $(\zeta_{n-j})_{l_1 l_2 \dots l_n}$ is the $n \times n \times \dots \times n$ matrices whose non-vanishing components are given by

$$(\zeta_{n-j})_{kn \ kn \ (k-1)n+1 \dots \widehat{(k-1)n+i} \dots kn-1} = \delta_{n-i \ n-j+1}. \quad (3.19)$$

Here, i and j run from 2 to $n - 1$ and ζ_{n-j} s are symmetric with respect to permutations among indices $\{kn \ kn \ (k-1)n+1 \dots \widehat{(k-1)n+i} \dots kn-1\}$ for $k = 1, \dots, n$. We find that the time-dependent components in $V_\alpha(t)$ s yield the equation of motion of harmonic oscillators under the Hamiltonians whose non-vanishing components are given by

$$(H_1)_{kn \ kn \ (k-1)n+1 \dots kn-2} = (-1)^{k(n-1)} \hbar^{(n)} \Omega, \quad (3.20)$$

$$(H_B)_{kn \ kn \ (k-1)n+1 \dots \widehat{(k-1)n+i} \dots kn-1} = \delta_{n-i \ B}, \quad (3.21)$$

where B runs from 2 to $n - 1$ and k runs from 1 to n . The H_A s are symmetric with respect to permutations among indices. The $(V_\alpha(t))_{l_1 l_2 \dots l_n}$ s satisfy the following algebra:

$$[V_1(t), V_2(t), \dots, V_n(t)]_{l_1 l_2 \dots l_n} = -i \hbar^{(n)} (J^{(N)})_{l_1 l_2 \dots l_n}, \quad (3.22)$$

where $(J^{(N)})_{l_1 l_2 \dots l_n}$ is the real normal $n^2 \times n^2 \times \dots \times n^2$ matrices whose non-vanishing components are given by

$$(J^{(N)})_{kn \ kn \ (k-1)n+1 \dots kn-2} = (-1)^{k(n-1)}. \quad (3.23)$$

In both cases, the $n + 1$ variables, $(\xi(t), \eta(t), H_A)$ or $(V_\alpha(t), J^{(N)})$, form a closed algebra for the n -fold commutator, which is regarded as a generalization of spin algebra^{9), 8), 6), 10)}.

§4. Conclusions

We have given our definition of n -index objects (n -th power matrices) and their algebraic properties, and formulated a dynamical system based on hermitian n -th power matrices. The basic structure of our mechanics is summarized as follows. For hermitian n -th power matrices $(V_\alpha(t))_{l_1 l_2 \dots l_n} = (V_\alpha)_{l_1 l_2 \dots l_n} e^{i \Omega_{l_1 l_2 \dots l_n} t}$, their time evolution is regarded as the symmetry transformation generated by the Hamiltonians $(H_A)_{l_1 l_2 \dots l_n}$, such that $i \hbar^{(n)} \delta(V_\alpha(t))_{l_1 l_2 \dots l_n} = [V_\alpha(t), H_1, \dots, H_{n-1}]_{l_1 l_2 \dots l_n} \delta t$, which is the generalization of the Heisenberg's equation. The Hamiltonians $(H_A)_{l_1 l_2 \dots l_n}$ are

real normal forms where $(H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n}$ satisfies n -cocycle condition. There is the relation among $\Omega_{l_1 l_2 \cdots l_n}$ and H_A s such that $\hbar^{(n)} \Omega_{l_1 l_2 \cdots l_n} = -(H_1 \cdots \widetilde{H_{n-1}})_{l_1 l_2 \cdots l_n}$. An arbitrary normal n -th power matrix, $A_{l_1 l_2 \cdots l_n}^{(N)}$, is a constant of motion; i.e., $i\hbar^{(n)} dA_{l_1 l_2 \cdots l_n}^{(N)} / dt = [A^{(N)}, H_1, \cdots, H_{n-1}]_{l_1 l_2 \cdots l_n} = 0$. There are simple systems of harmonic oscillators described by hermitian n -th power matrices.

Our mechanics is regarded as the generalization of Heisenberg's matrix mechanics because it reduces to Heisenberg's matrix mechanics in case with $n = 2$. In quantum mechanics, a matrix element $A_{l_1 l_2}$ is interpreted as a probability amplitude between the state labeled by l_1 and that labeled by l_2 . A similar interpretation for an n -th power matrix element ($n \geq 3$), however, is not yet known, and it is not clear whether many-index objects is applicable to real physical systems.*⁾ It would be worth while to explore physical meaning of many-index objects based on generalized spin variables.

Acknowledgements

This work was supported in part by Scientific Grants from the Ministry of Education and Science, Grant No. 13135217 and Grant No. 15340078.

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*⁾ In Ref. 10), a generalization of spin algebra using three-index objects has been proposed and the connection between a triple commutation relation and an uncertainty relation has been discussed.